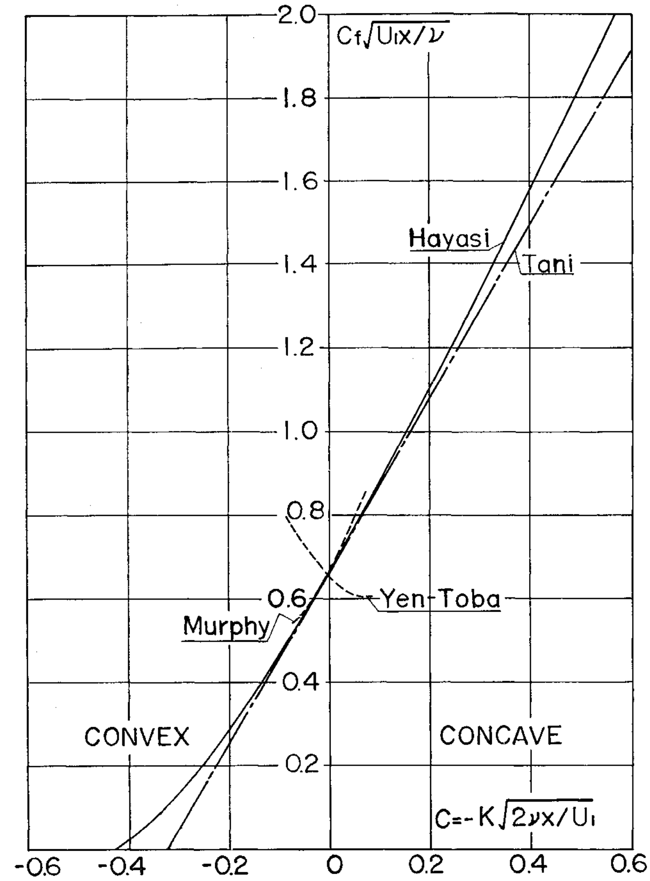

Fig. 1 Skin friction coefficient as a function of surface curvature

1. From Eqs. (38) and (52) of Yen and Toba's paper, one has

$$n = -K^{-1}(1 - e^{-C\chi}) \tag{3}$$

where n is the distance along a curvilinear coordinate line. Therefore one obtains, from their Eqs. (46) and (A-1), $v_\xi = U_1 f'/(1 + Kn)$, where v_ξ is the velocity. In potential flow, one gets from their Eq. (60)

$$u_{\text{pot}} = U_1/(1 + Ky) \tag{4}$$

where y is the distance normal to the surface. Since $v_\xi \doteq u$ and $n \doteq y$, one has

$$f' = \phi' = u/u_{\text{pot}} \tag{5}$$

Therefore ϕ' is the ratio of the velocity in the boundary-layer flow to that in the potential flow on the same normal. Expressing Murphy's f as F, one has, from Eqs. (4) and (5) and Murphy's Eq. (30), $dF/d\eta = 2f'/(1 + 2A\eta)$. Since $A \doteq -2^{-1/2}C$, one obtains from Eq. (3)

$$\chi = 2^{-1/2}A^{-1}\ln(1 + 2A\eta) \tag{6}$$

Thus one has $F = 2^{1/2}f$. Therefore Murphy's Eq. (32) reduces to the present Eq. (2), which was derived from Yen and Toba's equation. Equation (6) shows that $\chi = 0$ at $\eta = 0$ and that $\chi \rightarrow \infty$ when $\eta \rightarrow \infty$ for $A \geq 0$, or when $\eta \rightarrow -(2A)^{-1}$ for $A < 0$. Therefore, the boundary conditions for F also reduce to those for ϕ. Thus it is clear that Murphy's and Yen's analyses are essentially the same, and both are correct.

For the purpose of numerical calculation, Eq. (2) is very convenient. Indeed, with the transformations $\phi = ky$, $\chi = k^{-1}X$, one has $y''' + yy'' = 0$, together with boundary conditions $y(0) = 2Ck^{-1}$, $y'(0) = 0$, $y'(\infty) = k^{-2}$. Thus, if a solution y', satisfying the initial conditions $y(0) = \alpha$, $y'(0) = 0$, $y''(0) = \beta$, tends to γ when $X \rightarrow \infty$, then one can obtain a solution of Eq. (2) from the initial conditions $\phi(0) = 2C = \alpha\gamma^{-1/2}$, $\phi'(0) = 0$, and $\phi''(0) = \beta\gamma^{-3/2}$. In such a way, numerical solutions of Eq. (2) have been obtained on an electronic digital computer Datatron 205. The integration was done by using the Runge-Kutta method with fourth-order accuracy. The interval of calculation is 0.01. The variation of C_f is plotted in Fig. 1, where the results of Murphy and Yen and Toba and the line representing Tani's formula[1]

$$C_f = 0.664 + 2.05C \tag{7}$$

also are presented. In the range $-0.1 < C < 0.1$, Tani's formula is in good agreement with the present result. The deviation of Murphy's result from the present one may be attributed to his questionable use of series expansion for the determination of C_f. Yen and Toba's result seems to be erroneous. It may be suspected that they started their numerical calculation from a point too near to the surface, where their equation has a singularity, and that they used integration steps that are too large for treating such an equation.

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Some Physical Interpretations of Magnetohydrodynamic Duct Flows

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THIS note presents some physical interpretations of magnetohydrodynamic duct flows with various boundary conditions viewed in the light of the effects of conducting walls on the pattern of electric current, taking examples from published results on rectangular ducts.[1–4] The current patterns are illustrated in Fig. 1 for rectangular ducts having various combinations of conducting and nonconducting walls, a uniform magnetic field being applied in the horizontal direction.

There is an essential difference between the roles played by horizontal and vertical conducting walls. A horizontal conducting wall serves only as an electrode (cases A[1] and B[2]) or as a short cut for the current (case D[2]). Therefore the mechanism of flow resistance in case D remains essentially the same as in a duct of nonconducting walls (case E[3]). On the contrary, a vertical conducting wall acts essentially to pass the current in the vertical direction outside of the fluid, thus resulting in a net current in the fluid which makes a primary contribution to flow resistance at large Hartmann number $M = B_0a(\sigma/\eta)^{1/2}$.

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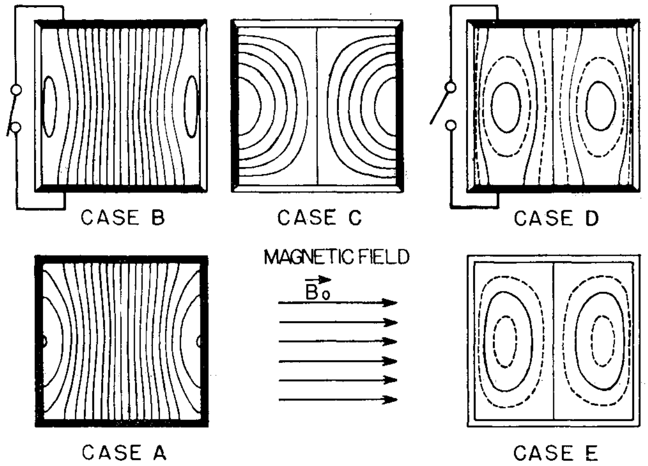

Fig. 1 Patterns of electric current for constant flow rate in square ducts with conducting (black) or nonconducting (white) walls. Hartmann number $M = B_0a(\sigma/\eta)^{1/2} = 2$, where B_0, a, σ, and η denote flux density of external magnetic field, half the length of a side of duct, conductivity, and viscosity of fluid, respectively. The horizontal walls are short-circuited in case B but unconnected in case D

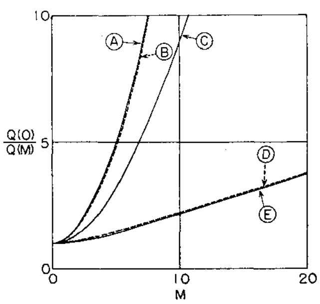

Fig. 2 Reciprocal of rate of flow vs Hartmann number. $Q(M)$ denotes the volume rate of flow at Hartmann number M, and $Q(0)$ that at $M = 0$; (A) Chang and Lundgen,[1] (B) Tani,[2] (C) Lundgen, Atabeck, and Chang,[4] (D) Tani,[2] and (E) Shercliff[3]

As a matter of fact, there is a strong resemblance between cases A and B both in current pattern and flow rate (Figs. 1 and 2). Indeed, only 21 and 16% of the current (this percentage tending to zero as M or the height/breadth ratio of the duct becomes infinitely large) comes from the vertical walls in case A at Hartmann number $M = 5$ and 10, respectively. (Thus Hartmann flow with conducting walls[1] is exactly the same as with nonconducting walls with net current.)

Although the current patterns in cases D and E are different in the neighborhood of the horizontal walls, the flow rates are expected to be only slightly different when $M^{1/2}$ is large, because the electric resistance to the current loop in the boundary layer on the horizontal wall of case E is of the order of $M^{1/2}$, whereas the principal contribution of the order of M comes from the boundary layer on the vertical wall in both cases D and E.

Finally, case C[4] has the same mechanism of flow resistance as cases A and B at large M, but the current loop in C suffers electric resistance of the order of $M^{1/2}$ in the boundary layer on the horizontal wall. Thus the flow rate for this case at large M may be expected to be similar to case A with the conductivity of the fluid somewhat decreased.

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Unification of Matrix Methods of Structural Analysis

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Nomenclature

S = column matrix of internal stresses
b_1 = transformation matrix of unknowns into internal stresses
b_0 = transformation matrix of external loads into internal stresses
f = flexibility matrix of individual elements
R = column matrix of external loads
w = joint displacements
X = unknowns (forces or joint displacements)
a = transformation matrix of joint displacements into strains
r = stiffness matrix of individual elements
D = $b_1{}^tfb_1$
D_0 = $b_1{}^tfb_0$
C = a^tra
0 = null matrix
t = transpose of a matrix

Introduction

The matrix methods of structural analysis which have appeared recently in the literature are of two classes: the Argyris method and the Klein method. The Argyris method can be subdivided into those in which forces or deformations are taken as unknowns. This is stated clearly in Ref. 1, where the basic references of the two methods are mentioned.

In Ref. 2, Klein exposes the foundations of his method and points out as one of his disadvantages: "Matrix is large." Later, in more recent works,[3,4] Klein advocates a pretriangularization of his initial equations to avoid that disadvantage. He also states that the ideal pretriangularization is obtained when the redundant part of the structural system is isolated. In this case the order of the matrix which has to be inverted is much less than the order of the initial large matrix.

The purpose of this note is to show that Argyris' equations are exactly Klein's after the ideal pretriangularization is obtained. This conclusion allows the unification of all methods of matrix structural analysis.

Theory

By the Argyris formulation, taking forces as unknowns, the internal stresses and the joint displacements of a structure submitted to external loads applied at the joints are, respectively,[5]

$$S = b_0R + b_1X \tag{1}$$

and

$$w = b_0{}^tfS \tag{2}$$

where X is given by

$$DX + D_0R = 0 \tag{3}$$

Equations (1–3) can be written jointly in Table 1, where (b_1) and (b_0) are, respectively, matrices b_1 and b_0 in which the rows corresponding to the independent internal stresses are excluded, and $(b_0{}^tf)$ is matrix $b_0{}^tf$ rearranged so that columns referring to the independent internal stresses are written first.

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